Complex Numbers

Circular functions – Powers, multiple angles, series

1. $2\cos\theta = x + \dfrac{1}{x} \Rightarrow x = \cos\theta \pm i\sin\theta \Rightarrow x^n = (\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$, by de Moivre's Theorem.

$x^n + \dfrac{1}{x^n} = (\cos n\theta \pm i\sin n\theta) + (\cos n\theta \mp i\sin n\theta) = 2\cos n\theta$.

$5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0 \quad \Rightarrow \quad 5\left(x^2 + \dfrac{1}{x^2}\right) - 11\left(x + \dfrac{1}{x}\right) + 16 = 0 \Rightarrow 10\cos 2\theta - 22\cos\theta + 16 = 0$

$\therefore \quad 10\,(2\cos^2\theta - 1) - 22\cos\theta + 16 = 0 \quad \Rightarrow \quad 20\cos^2\theta - 22\cos\theta + 6 = 0 \quad \Rightarrow \quad 10\cos^2\theta - 11\cos\theta + 3 = 0$

$\therefore \quad \cos\theta = \dfrac{1}{2}$ or $\dfrac{3}{5}$. The corresponding $\sin\theta = \dfrac{\sqrt{3}}{2}$ or $\dfrac{4}{5}$ (draw triangles to show)

Hence $x = \cos\theta \pm i\sin\theta = \dfrac{1 \pm \sqrt{3}i}{2}, \ \dfrac{3 \pm 4i}{5}$.

2. $u = \cos\dfrac{2}{7}\pi + i\sin\dfrac{2}{7}\pi \quad \Rightarrow u^7 = \left(\cos\dfrac{2}{7}\pi + i\sin\dfrac{2}{7}\pi\right)^7 = \cos 2\pi + i\sin 2\pi = 1$

$\therefore \quad u^7 - 1 = 0$, and the roots can be written $1, u, u^2, u^3, u^4, u^5, u^6$.

Sum of roots $= 1 + u + u^2 + u^3 + u^4 + u^5 + u^6 = 0$.

If $\alpha = u + u^2 + u^4$, $\beta = u^3 + u^5 + u^6$, then $\alpha + \beta = u + u^2 + u^3 + u^4 + u^5 + u^6 = -1$

$\alpha\beta = u + u^2 + u^3 + u^4 + u^5 + u^6 + 3 = -1 + 3 = 0$.

It follows that α, β are roots of the equation $x^2 + x + 2 = 0$.

Solve this equation and take $\alpha = -\dfrac{1}{2} + \dfrac{\sqrt{7}}{2}i$ …. (1)

On the other hand, $\alpha = u + u^2 + u^4 = \left(\cos\dfrac{2}{7}\pi + i\sin\dfrac{2}{7}\pi\right) + \left(\cos\dfrac{4}{7}\pi + i\sin\dfrac{4}{7}\pi\right) + \left(\cos\dfrac{8}{7}\pi + i\sin\dfrac{8}{7}\pi\right)$ …. (2)

Equate real and imaginary parts, $\cos\dfrac{2}{7}\pi + \cos\dfrac{4}{7}\pi + \cos\dfrac{8}{7}\pi = -\dfrac{1}{2}$ and $\sin\dfrac{2}{7}\pi + \sin\dfrac{4}{7}\pi + \sin\dfrac{8}{7}\pi = \dfrac{1}{2}\sqrt{7}$

3. **(i)** $\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$= c^7 + 7c^6(is) + 21c^5(is)^2 + 35c^4(is)^3 + 35c^3(is)^4 + 21c^2(is)^5 + 7c(is)^6 + (is)^7$, where $c = \cos\theta$, $s = \sin\theta$

$= (c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6) + i\,(7c^6s - 35c^4s^3 + 21c^2s^5 - s^7)$

Compare imaginary parts,

$\sin 7\theta = 7c^6s - 35c^4s^3 + 21c^2s^5 - s^7 = 7(1 - s^2)^3s - 35(1 - s^2)^2s^3 + 21(1 - s^2)s^5 - s^7$

$= 7(1 - 3s^2 + 3s^4 - s^6)s - 35(1 - 2s^2 + s^4)s^3 + 21(1 - s^2)s^5 - s^7 = 7s - 56s^3 + 112s^5 - 64s^7$

$\therefore \quad \sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

(ii) Let $z = \cos\theta + i\sin\theta$, then $1/z = \cos\theta - i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$, $1/z^n = \cos n\theta - i\sin n\theta$

$\therefore \quad \cos n\theta = \dfrac{1}{2}\left(z^n + \dfrac{1}{z^n}\right), \quad \sin n\theta = \dfrac{1}{2i}\left(z^n - \dfrac{1}{z^n}\right)$

$$64\sin^7\theta = 64\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^7 = 64\times\frac{1}{-128\,i}\left[z^7 - 7z^5 + 21z^3 - 35z + \frac{35}{z} - \frac{21}{z^3} + \frac{7}{z^5} - \frac{1}{z^7}\right]$$

$$= -\left[\frac{1}{2i}\left(z^7 - \frac{1}{z^7}\right) - 7\times\frac{1}{2i}\left(z^5 - \frac{1}{z^5}\right) + 35\times\frac{1}{2i}\left(z^3 - \frac{1}{z^3}\right)\right]$$

$= -[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta] = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta$

(iii) $$2^6\sin^5\theta\cos^2\theta = 64\times\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^5\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2$$

$$= 64\times\frac{1}{32i}\left[z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}\right]\times\frac{1}{4}\left[z^2 + 2 + \frac{1}{z^2}\right]$$

$$= \frac{1}{2i}\left[z^7 - 3z^5 + z^3 + 5z - \frac{5}{z} - \frac{1}{z^3} + \frac{3}{z^5} - \frac{1}{z^7}\right] = \frac{1}{2i}\left(z^7 - \frac{1}{z^7}\right) - 3\times\frac{1}{2i}\left(z^5 - \frac{1}{z^5}\right) + \frac{1}{2i}\left(z^3 - \frac{1}{z^3}\right) + 5\times\frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$= \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta$

(iv) $$64\left(\cos^8\theta + \sin^8\theta\right) = 64\times\left\{\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^8 + \left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^8\right\}$$

$$= \frac{1}{4}\times\left\{\left[z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + \frac{56}{z^2} + \frac{28}{z^4} + \frac{8}{z^2} + \frac{1}{z^8}\right] + \left[z^8 - 8z^6 + 28z^4 - 56z^2 + 70 - \frac{56}{z^2} + \frac{28}{z^4} - \frac{8}{z^2} + \frac{1}{z^8}\right]\right\}$$

$$= \frac{1}{2}\times\left[z^8 + 28z^4 + 70 + \frac{28}{z^4} + \frac{1}{z^8}\right] = \frac{1}{2}\left(z^8 + \frac{1}{z^8}\right) + 28\times\frac{1}{2}\left(z^4 + \frac{1}{z^4}\right) + 35$$

$= \cos 8\theta + 28\cos 4\theta + 35$

4. $p = \text{cis}\,\alpha$, $q = \text{cis}\,\beta$, $r = \text{cis}\,\gamma$,

If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$ then $p + q + r = 0$.

Since $p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) = 0$, we have $p^3 + q^3 + r^3 = 3pqr$.

By de Moivre's Theorem, $p^3 = (\text{cis}\,\alpha)^3 = \text{cis}\,3\alpha$, $q^3 = (\text{cis}\,\beta)^3 = \text{cis}\,3\beta$, $r^3 = (\text{cis}\,\gamma)^3 = \text{cis}\,3\gamma$

and $3pqr = 3\,\text{cis}\,(\alpha + \beta + \gamma)$

Equating real parts and imaginary parts , we get $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

5. $$2\cos x = u + \frac{1}{u},\quad 2\cos y = v + \frac{1}{v}\quad \Rightarrow u = \cos x \pm i\sin x,\qquad v = \cos y \pm i\sin y$$

$$\Rightarrow u^m = (\cos x \pm i\sin x)^m = \cos mx \pm i\sin mx,\qquad v^n = (\cos y \pm i\sin y)^n = \cos ny \pm i\sin ny$$

$$\Rightarrow u^m v^n = (\cos mx \pm i\sin mx)(\cos ny \pm i\sin ny) = \cos(m+n)x \pm i\sin(m+n)x$$

$$\Rightarrow u^m v^n + \frac{1}{u^m v^n} = [\cos(m+n)x \pm i\sin(m+n)x] + [\cos(m+n)x \mp i\sin(m+n)x] = 2\cos(mx + ny)$$

6. $\cos 9\theta + i\sin 9\theta = (\cos\theta + i\sin\theta)^9$

$= c^9 + 9c^8(is) + 36c^7(is)^2 + 84c^6(is)^3 + 126c^5(is)^4 + 126c^4(is)^5 + 84c^3(is)^6 + 36c^2(is)^7 + 9c(is)^8 + (is)^9$, $c = \cos\theta$, $s = \sin\theta$

$= (c^9 - 36c^7s^2 + 126c^5s^4 - 84c^3s^6 + 9cs^8) + i(9c^8s - 84c^6s^3 + 126c^4s^5 - 36c^2s^7 + s^9)$

$$\tan 9\theta = \frac{\sin 9\theta}{\cos 9\theta} = \frac{9c^8s - 84c^6s^3 + 126c^4s^5 - 36c^2s^7 + s^9}{c^9 - 36c^7s^2 + 126c^5s^4 - 84c^3s^6 + 9cs^8} = \frac{9x - 84x^3 + 126x^5 - 36x^7 + x^9}{1 - 36x^2 + 126x^4 - 84x^6 + 9x^8}$$ **,** where $x = \tan\theta$ …. (1)

$\therefore\ \tan 9\theta = 0 \qquad \Rightarrow 9\theta = r\pi \pm 0 \quad \Rightarrow \theta = \frac{r\pi}{9}$, where $r = 0, 1, 2, 3, \dots, 8$. …. (2)

From (1), $\tan 9\theta = 0 \quad \Rightarrow \quad 9x - 84x^3 + 126x^5 - 36x^7 + x^9 = 0$

$\Rightarrow \quad x\left(9 - 84x^2 + 126x^4 - 36x^6 + x^8\right) = 0$, where $x = \tan\theta$ …. (3)

$\therefore\ \tan\frac{r\pi}{9}$, where $r = 0, 1, 2, 3, \dots, 8$, are roots of the equation $x\left(9 - 84x^2 + 126x^4 - 36x^6 + x^8\right) = 0$

$\therefore\ \tan\frac{r\pi}{9}$, where $r = 1, 2, 3, \dots, 8$, are roots of the equation $9 - 84x^2 + 126x^4 - 36x^6 + x^8 = 0$ …. (4)

Now, put $y = x^2$,

$\tan^2\frac{r\pi}{9}$, where $r = 1, 2, 3, 4, \dots, 8$ are the roots of the equation $9 - 84y + 126y^2 - 36y^3 + y^4 = 0$ …. (5)

or $\tan^2\frac{r\pi}{9}$, where $r = 1, 2, 3, 4, -1, -2, -3, -4$ are the roots of the equation (5)

But $\tan^2\frac{r\pi}{9} = \tan^2\left(-\frac{r\pi}{9}\right)$, $\therefore$ $\tan^2\frac{r\pi}{9}$, where $r = 1, 2, 3, 4$ are the roots of the equation (5) .

By Vieta Theorem, Sum of roots = $\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{3\pi}{9} + \tan^2\frac{4\pi}{9}$ = - coefficient of y^3 in (5) = 36

Product of roots = $\tan^2\frac{\pi}{9}\tan^2\frac{2\pi}{9}\tan^2\frac{3\pi}{9}\tan^2\frac{4\pi}{9}$ = constant term = 9 $\Rightarrow$ $\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{3\pi}{9}\tan\frac{4\pi}{9} = 3$

(negative root is rejected)

7. By the Binomial Theorem,

$$(1+i)^{2n} = \binom{2n}{0} + \binom{2n}{1}i + \binom{2n}{2}i^2 + \binom{2n}{3}i^3 + \dots + \binom{2n}{2n-1}i^{2n-1} + \binom{2n}{2n}i^{2n}$$

$$= \sum_{k=0}^{n}\binom{2n}{2k}i^{2k} + \sum_{k=0}^{n-1}\binom{2n}{2k+1}i^{2k+1} = \sum_{k=0}^{n}(-1)^k\binom{2n}{2k} + i\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}i^{2k+1} \qquad \dots.\ (1)$$

Also, by de Moivre's Theorem,

$$(1+i)^{2n} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{2n} = 2^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) \qquad \dots.\ (2)$$

By equating real and imaginary parts of (1) and (2),

(i) $\sum_{k=0}^{n}(-1)^k\binom{2n}{2k} = 2^n\cos\frac{n\pi}{2}$ **(ii)** $\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1} = 2^n\sin\frac{n\pi}{2}$.

8. $z = 1 + \cos\theta + i\sin\theta$, $|z - 1| = |1 + \cos\theta + i\sin\theta - 1| = |\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$

$\therefore$ satisfies $|z - 1| = 1$, where $-\pi \le \theta \le \pi$.

$$\frac{1}{z} = \frac{1}{1+\cos\theta + i\sin\theta} = \frac{1+\cos\theta - i\sin\theta}{(1+\cos\theta)^2 + \sin^2\theta} = \frac{1+\cos\theta + i\sin\theta}{2(1+\cos\theta)} = \frac{1}{2} + i\frac{\sin\theta}{2(1+\cos\theta)}$$

$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{2}$, a constant. $\therefore$ $\frac{1}{z}$ describes a straight line $x = \frac{1}{2}$ in the Argand diagram.

$$z = 1 + \cos\theta + i\sin\theta = 2\cos^2\frac{\theta}{2} + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \qquad -\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{\pi}{2}$$

$\therefore \quad |z| = 2\cos\frac{\theta}{2}, \qquad \arg(z) = \frac{\theta}{2}$

$(1 + \cos\theta + i\sin\theta)^n$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) = \left(2^n\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}\right) + i\left(2^n\cos^n\frac{\theta}{2}\sin\frac{n\theta}{2}\right) \quad \ldots\ (1)$$

$$(1 + \cos\theta + i\sin\theta)^n = (1+\omega)^n = \sum_{r=0}^{n} C_r^n\omega^r = \sum_{r=0}^{n} C_r^n(\cos\theta + i\sin\theta)^r = \sum_{r=0}^{n} C_r^n(\cos r\theta + i\sin r\theta) \quad \ldots\ (2)$$

Comparing real part of (1) and (2), $\quad 1 + C_1^n\cos\theta + C_2^n\cos 2\theta + \ldots + C_n^n\cos n\theta = \left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$.

9. Consider $C = \sum_{r=1}^{n}\binom{n}{r}\cos 2r\theta \qquad S = \sum_{r=1}^{n}\binom{n}{r}\sin 2r\theta$

$$\therefore \quad C + iS = \sum_{r=1}^{n}\binom{n}{r}(\cos 2r\theta + i\sin 2r\theta) = \sum_{r=1}^{n}\binom{n}{r}(\cos 2\theta + i\sin 2\theta)^r = (1 + \cos 2\theta + i\sin 2\theta)^n$$

$$= (2\cos^2\theta + i2\sin\theta\cos\theta)^n = (2\cos\theta)^n(\cos\theta + i\sin\theta)^n = (2\cos\theta)^n(\cos n\theta + i\sin n\theta)$$

Comparing imaginary parts, $\quad \sum_{r=1}^{n}\binom{n}{r}\sin 2r\theta = 2^n\cos^n\theta\ \sin n\theta$.

10. **(i)** Let $\quad C = x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \ldots + x^{n-1}\cos(n-1)\theta$

$S = x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \ldots + x^{n-1}\sin(n-1)\theta$

$\therefore \quad C + iS = x\,\mathrm{cis}\,\theta + x^2\,\mathrm{cis}\,2\theta + x^3\,\mathrm{cis}\,3\theta + \ldots + x^{n-1}\mathrm{cis}\,(n-1)\theta$, where $\mathrm{cis}\,\theta = \cos\theta + i\sin\theta$.

$= x\,\mathrm{cis}\,\theta + x^2(\mathrm{cis}\,\theta)^2 + x^3(\mathrm{cis}\,\theta)^3 + \ldots + x^{n-1}(\mathrm{cis}\,\theta)^{n-1}$, by de Moivre's Theorem

$= x\,\mathrm{cis}\,\theta + (x\mathrm{cis}\,\theta)^2 + (x\mathrm{cis}\,\theta)^3 + \ldots + (x\mathrm{cis}\,\theta)^{n-1}$

$$= x\mathrm{cis}\theta\frac{(x\mathrm{cis}\theta)^{n-1} - 1}{x\mathrm{cis}\theta - 1} = \frac{(x\mathrm{cis}\theta)^n - x\mathrm{cis}\theta}{x\mathrm{cis}\theta - 1} = \frac{x^n\mathrm{cis}n\theta - x\mathrm{cis}\theta}{x\mathrm{cis}\theta - 1} = \frac{(x^n\cos n\theta - x\cos\theta) + i(x^n\sin n\theta - x\sin\theta)}{(x\cos\theta - 1) + i(x\sin\theta)}$$

$$= \frac{(x^n\cos n\theta - x\cos\theta) + i(x^n\sin n\theta - x\sin\theta)}{(x\cos\theta - 1)^2 + (x\sin\theta)^2}[(x\cos\theta - 1) - i(x\sin\theta)]$$

Comparing imaginary parts, $\quad S = \frac{(x^n\sin n\theta - x\sin\theta)(x\cos\theta - 1) - (x\sin\theta)(x^n\cos n\theta - x\cos\theta)}{(x\cos\theta - 1)^2 + (x\sin\theta)^2}$

$$= \frac{x^{n+1}\sin n\theta\cos\theta - x^n\sin n\theta + x\sin\theta - x^{n+1}\sin\theta\cos n\theta}{x^2 - 2x\cos\theta} = \frac{x^n\sin n\theta\cos\theta - x^{n-1}\sin n\theta + \sin\theta - x^n\sin\theta\cos n\theta}{x - 2\cos\theta}$$

(ii) Let $\quad C = \sum_{r=0}^{n-1}(r+1)\cos r\theta\ , \quad S = \sum_{r=0}^{n-1}(r+1)\sin r\theta$

$$\therefore \quad E = C + iS = \sum_{r=0}^{n-1}(r+1)\mathrm{cis}\ r\theta = \sum_{r=0}^{n-1}(r+1)(\mathrm{cis}\ \theta)^r \quad \ldots\ (1)$$

$$\therefore \quad (\mathrm{cis}\ \theta)E = \sum_{r=0}^{n-1}(r+1)(\mathrm{cis}\ \theta)^{r+1} = \sum_{r=1}^{n} r(\mathrm{cis}\ \theta)^r \quad \ldots\ (2)$$

(2) – (1), $\quad [(\mathrm{cis}\ \theta) - 1]E = n(\mathrm{cis}\ \theta)^n + \sum_{r=1}^{n-1} r(\mathrm{cis}\ \theta)^r - 1 = n(\mathrm{cis}\ n\theta) + (\mathrm{cis}\ \theta)\frac{(\mathrm{cis}\ \theta)^{n-1} - 1}{(\mathrm{cis}\ \theta) - 1} - 1$

$$\therefore \quad E = \frac{n(\text{cis } n\theta)(\text{cis } \theta) + (\text{cis } \theta)^n - (\text{cis } \theta) - (\text{cis } \theta) + 1}{[(\text{cis } \theta) - 1]^2} = \frac{n \ \text{cis } [(n+1)\theta] + (\text{cis } n\theta) - 2(\text{cis } \theta) + 1}{\left(\text{cis } \frac{\theta}{2}\right)^2 \left[\text{cis } \frac{\theta}{2} - \text{cis}\left(-\frac{\theta}{2}\right)\right]^2}$$

$$= \frac{n \ \text{cis } [(n+1)\theta] + (\text{cis } n\theta) - 2(\text{cis } \theta) +}{(\text{cis } \theta)\left(2i \sin \frac{\theta}{2}\right)^2} = -\frac{n\,(\text{cis } n\theta) + \text{cis } (n-1)\theta - 2 + \text{cis}(-\theta)}{4 \sin^2 \frac{\theta}{2}}$$

Compare imaginary parts, we have $S = \; = -\dfrac{n\,(\sin n\theta) + \sin\,(n-1)\theta - 2 - \sin\theta}{4\sin^2 \frac{\theta}{2}}$.

(iii) Let $C = \sum_{r=1}^{n} \sin^r \theta \ \cos r\theta$, $S = \sum_{r=1}^{n} \sin^r \theta \ \sin r\theta$

$$\therefore \quad C + iS = \sum_{r=1}^{n} \sin^r \theta \ \text{cis } r\theta = \sum_{r=1}^{n} \sin^r \theta \ (\text{cis } \theta)^r = \sum_{r=1}^{n} [\sin\theta \ (\text{cis } \theta)]^r = [\sin\theta \ (\text{cis } \theta)] \frac{[\sin\theta \ (\text{cis } \theta)]^n - 1}{[\sin\theta \ (\text{cis } \theta)] - 1}$$

$$= \frac{[\sin\theta \ (\text{cis } \theta)]^{n+1} - \sin\theta \ (\text{cis } \theta)}{(\sin\theta\cos\theta - 1) + i(\sin^2\theta)} = \frac{\sin^{n+1}\theta \ \text{cis}[(n+1)\theta] - \sin\theta \ (\text{cis } \theta)}{(\sin\theta\cos\theta - 1) + i(\sin^2\theta)}$$

$$= \frac{\{\sin^{n+1}\theta \ \cos[(n+1)\theta] - \sin\theta \ \cos\theta\} + i\{\sin^{n+1}\theta \ \sin[(n+1)\theta] - \sin^2\theta\}}{(\sin\theta\cos\theta - 1)^2 + (\sin\theta)^4}\{(\sin\theta\cos\theta - 1) - i(\sin^2\theta)\}$$

Comparing imaginary parts,

$$S = \frac{(\sin\theta\cos\theta - 1)\{\sin^{n+1}\theta \ \sin[(n+1)\theta] - \sin^2\theta\} - (\sin^2\theta)\{\sin^{n+1}\theta \ \cos[(n+1)\theta] - \sin\theta \ \cos\theta\}}{\sin^2\theta\cos^2\theta - 2\sin\theta\cos\theta + 1 - \sin^4\theta}$$

$$= \frac{\sin^{n+2}\theta\cos\theta\sin[(n+1)\theta] - \sin^{n+1}\theta\sin[(n+1)\theta] + \sin^2\theta - \sin^{n+3}\theta\cos[(n+1)\theta]}{\sin^2\theta(1 - \sin^2\theta) - 2\sin\theta\cos\theta + 1 - \sin^4\theta}$$

$$= \frac{\sin^{n+2}\theta\{\cos\theta\sin[(n+1)\theta] - \sin\theta\cos[(n+1)\theta]\} - \sin^{n+1}\theta\sin[(n+1)\theta] + \sin^2\theta}{\sin^2\theta - 2\sin\theta\cos\theta + 1}$$

$$= \frac{\sin^{n+2}\theta\sin n\theta - \sin^{n+1}\theta\sin[(n+1)\theta] + \sin^2\theta}{\sin^2\theta - 2\sin\theta\cos\theta + 1}$$

11. $C = \cos(2n+1)\theta - C_1^{2n+1}\cos(2n-1)\theta + \ldots + (-1)^r C_r^{2n+1}\cos(2n+1-2r)\theta + \ldots + (-1)^n C_n^{2n+1}\cos\theta$

$S = \sin(2n+1)\theta - C_1^{2n+1}\sin(2n-1)\theta + \ldots + (-1)^r C_r^{2n+1}\sin(2n+1-2r)\theta + \ldots + (-1)^n C_n^{2n+1}\sin\theta$

$\therefore$ $C + iS = \text{cis}(2n+1)\theta - C_1^{2n+1}\text{cis}(2n-1)\theta + \ldots + (-1)^r C_r^{2n+1}\text{cis}(2n+1-2r)\theta + \ldots + (-1)^n C_n^{2n+1}\text{cis}\theta$

$$= (\text{cis}\theta)^{2n+1} - C_1^{2n+1}(\text{cis}\theta)^{2n-1} + \ldots + (-1)^r C_r^{2n+1}(\text{cis}\theta)^{2n+1-2r} + \ldots + (-1)^n C_n^{2n+1}\text{cis}\theta$$

$$= (\text{cis}\theta)^{2n+1} - C_1^{2n+1}(\text{cis}\theta)^{2n}[\text{cis}(-\theta)] + \ldots + (-1)^r C_r^{2n+1}(\text{cis}\theta)^{2n+1-r}[\text{cis}(-\theta)]^r + \ldots + (-1)^n C_n^{2n+1}(\text{cis}\theta)^{n+1}[\text{cis}(-\theta)]^n$$

$= \dfrac{1}{2}\sum_{r=0}^{2n+1} C_r^{2n+1}(\text{cis}\theta)^{2n+1-r}[\text{cis}(-\theta)]^r$, since $C_r^{2n+1} = C_{(2n+1)-r}^{2n+1}$ and

$$(\text{cis}\theta)^{2n+1-r}[\text{cis}(-\theta)]^r = (\text{cis}\theta)^r[\text{cis}(-\theta)]^{2n+1-r}\{(\text{cis}\theta)^{2n+1-2r}[\text{cis}(-\theta)]^{2r-(2n+1)}\} = (\text{cis}\theta)^r[\text{cis}(-\theta)]^{2n+1-r}$$

$$= \frac{1}{2}\left[\text{cis}\theta - \text{cis}(-\theta)\right]^{2n+1} = \frac{1}{2}\left[2i\sin\theta\right]^{2n+1} = \frac{1}{2}(-1)^n 2^{2n+1}\sin^{2n+1}\theta$$

Comparing imaginary parts, $S = (-1)^n 2^{2n}\sin^{2n+1}\theta$

12. If n is odd, let $n = 2t-1$, $t \in \mathbb{N}$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}c^k(is)^{n-k} \quad , \quad \text{where} \quad c = \cos\theta \,, \quad s = \sin\theta$$

$$= \sum_{k=0}^{2t-1}\binom{2t-1}{k}c^k(is)^{2t-1-k} = \sum_{m=1}^{t}\binom{2t-1}{2m-1}c^{2m-1}(is)^{2t-1-(2m-1)} + \sum_{m=0}^{t-1}\binom{2t-1}{2m}c^{2m}(is)^{2t-1-2m} \quad \text{, group odd and even terms.}$$

$$= \sum_{m=1}^{t}\binom{2t-1}{2m-1}c^{2m-1}\left(i^2\right)^{t-m}s^{2t-2m} + i\sum_{m=0}^{t-1}\binom{2t-1}{2m}c^{2m}\left(i^2\right)^{t-m-1}s^{2t-2m-1}$$

$$= \sum_{m=1}^{t}(-1)^{t-m}\binom{2t-1}{2m-1}c^{2m-1}s^{2t-2m} + i\sum_{m=0}^{t-1}(-1)^{t-m-1}\binom{2t-1}{2m}c^{2m}s^{2t-2m-1}$$

Comparing imaginary parts,

$$\sin n\theta = \sum_{m=0}^{t-1}(-1)^{t-m-1}\binom{2t-1}{2m}c^{2m}s^{2t-2m-1} = \sum_{m=0}^{t-1}(-1)^{t-m-1}\binom{2t-1}{2m}\left(1-s^2\right)^m s^{2t-2m-1}$$

$$= \sum_{m=0}^{t-1}(-1)^{t-m-1}\binom{2t-1}{2m}\left[\sum_{k=0}^{m}\binom{m}{k}\left(-s^2\right)^k\right]s^{2t-2m-1} = \sum_{m=0}^{t-1}(-1)^{t-m-1}\binom{2t-1}{2m}\left[\sum_{k=0}^{m}(-1)^k\binom{m}{k}s^{2k}\right]s^{2t-2m-1}$$

$$= \sum_{m=0}^{t-1}\sum_{k=0}^{m}(-1)^{t-m+k-1}\binom{2t-1}{2m}\binom{m}{k}s^{2t-2m+2k-1}$$

Put $L = m-k$, or $k = m-L$, $\sin n\theta = \sum_{m=0}^{t-1}\sum_{L=0}^{m}(-1)^{t-L-1}\binom{2t-1}{2m}\binom{m}{m-L}s^{2t-2L-1}$

Interchanging summation, we get

$$\sin n\theta = \sum_{L=0}^{t-1}\sum_{m=L}^{t-1}(-1)^{s-L-1}\binom{2t-1}{2m}\binom{m}{m-L}s^{2t-2L-1} = \sum_{L=0}^{s-1}(-1)^{t-L-1}\left[\sum_{m=L}^{t-1}\binom{2t-1}{2m}\binom{m}{m-L}\right]s^{2t-2L-1}$$

$$= \sum_{L=0}^{t-1}(-1)^{t-L-1}\left[\sum_{m=L}^{t-1}\binom{2t-1}{2m}\binom{m}{L}\right]s^{2t-2L-1} \quad , \quad \text{since} \quad \binom{m}{m-L} = \binom{m}{L} \,.$$

$$= \sum_{r=1}^{t-1}(-1)^{r-1}\left[\sum_{m=t-r}^{t-1}\binom{2t-1}{2m}\binom{m}{t-r}\right]s^{2r-1} \quad , \quad \text{where} \quad r = t-L \,.$$

$= b_1 s + b_3 s^3 + \ldots + b_n s^n$, where $s = \sin\theta$, and $b_1, b_3, \ldots, b_n$ are real numbers independent of θ .

$$\therefore \quad b_{2r-1} = (-1)^{r-1}\left[\sum_{m=t-r}^{t-1}\binom{2t-1}{2m}\binom{m}{t-r}\right] \qquad \text{.... (1)}$$

Put $r = 1$ in (1), $b_1 = (-1)^{1-1}\left[\sum_{m=t-1}^{t-1}\binom{2t-1}{2m}\binom{m}{t-1}\right] = \binom{2t-1}{2(t-1)}\binom{t-1}{t-1} = \binom{n}{n-1}\times 1 = n$

Put $r = t$ in (1), $b_n = (-1)^{t-1}\left[\sum_{m=0}^{t-1}\binom{2t-1}{2m}\binom{m}{0}\right] = (-1)^{\frac{n-1}{2}}\sum_{m=0}^{\frac{n-1}{2}}\binom{n}{2m}$, $n = 2t-1$. …. (2)

Consider $(1+x)^n = \sum_{m=0}^{n}\binom{n}{m}x^m$ …. (3)

Put x = 1 in (3), $2^n = \sum_{m=0}^{n} \binom{n}{m}$ …. (4)

Put x = -1 in (3), $0 = \sum_{m=0}^{n} \binom{n}{m}(-1)^{m-1}$ …. (5)

(4) + (5), $2^n = 2\sum_{m=0, m \text{ is even}}^{n} \binom{n}{m} = 2\sum_{m=0}^{\frac{n-1}{2}} \binom{n}{2m} \Rightarrow \sum_{m=0}^{\frac{n-1}{2}} \binom{n}{2m} = 2^{n-1}$ …. (6)

From (2) and (6), $b_n = (-1)^{\frac{n-1}{2}} 2^{n-1}$.

Now, $\sin n\theta = b_1 s + b_3 s^3 + \dots + b_n s^n = \sum_{r=1}^{n} b_r s^r$ .... (7)

Differentiate (7), $n \cos n\theta = c\sum_{r=1}^{n} r b_r s^{r-1}$, where $c = \cos\theta$. …. (8)

Differentiate (8) $-n^2 \sin n\theta = -s\sum_{r=1}^{n} r b_r s^{r-1} + c^2 \sum_{r=2}^{n} r(r-1) b_r s^{r-2} = -\sum_{r=1}^{n} r b_r s^r + (1-s^2)\sum_{r=2}^{n} r(r-1) b_r s^{r-2}$

$n^2 \sin n\theta = \sum_{r=1}^{n} r^2 b_r s^r - \sum_{r=2}^{n} r(r-1) b_r s^{r-2}$

From (7), $n^2 \sum_{r=1}^{n} b_r s^r = \sum_{r=1}^{n} r^2 b_r s^r - \sum_{r=2}^{n} r(r-1) b_r s^{r-2}$

$\therefore \sum_{r=2}^{n} r(r-1) b_r s^{r-2} = \sum_{r=1}^{n} r^2 b_r s^r - \sum_{r=1}^{n} n^2 b_r s^r$

$\sum_{r=1}^{n} (r+1)(r+2) b_{r+2} s^r = \sum_{r=1}^{n} (r^2 - n^2) b_r s^r$

Compare coefficients of s^r – term,

$\therefore b_{r+2} = \frac{r^2 - n^2}{(r+1)(r+2)} b_r$.

$b_3 = \frac{1-n^2}{6} b_1 = \frac{1-n^2}{6} n$